

Quantum Error Correction Code in the Hamiltonian Formulation

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Abstract

The Hamiltonian model of quantum error correction code in the literature is often constructed with the help of its stabilizer formalism. But there have been many known examples of nonadditive codes which are beyond the standard quantum error correction theory using the stabilizer formalism. In this paper, we suggest the other type of Hamiltonian formalism for quantum error correction code without involving the stabilizer formalism, and explain it by studying the Shor nine-qubit code and its generalization. In this Hamiltonian formulation, the unitary evolution operator at a specific time is a unitary basis transformation matrix from the product basis to the quantum error correction code. This basis transformation matrix acts as an entangling quantum operator transforming a separate state to an entangled one, and hence the entanglement nature of the quantum error correction code can be explicitly shown up. Furthermore, as it forms a unitary representation of the Artin braid group, the quantum error correction code can be described by a braiding operator. Moreover, as the unitary evolution operator is a solution of the quantum Yang–Baxter equation, the corresponding Hamiltonian model can be explained as an integrable model in the Yang–Baxter theory. On the other hand, we generalize the Shor nine-qubit code and articulate a topic called quantum error correction codes using Greenberger-Horne-Zeilinger states to yield new nonadditive codes and channel-adapted codes.

Key Words: Quantum Error Correction, Hamiltonian, GHZ State, the Shor Code

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1 Introduction

Quantum information and computation [1, 2] is a new interdisciplinary field combining (quantum) physics, (advanced) mathematics and (modern) computer science, and one of its great aims is to build a real quantum computer on which quantum algorithms can successfully run. Quantum error correction codes are exploited to protect quantum information from various kinds of noise and perform a large-scale quantum computation with imperfect quantum gates. They were originally invented by Shor, Steane, Calderbank, et al. [3, 4, 5, 6], for examples, the Shor nine-qubit code [3], the Steane seven-qubit code [5, 6], and the perfect five-qubit code saturating the quantum Hamming bound [7, 8]. Additive quantum codes analogous to classical linear codes can be described in the stabilizer formalism which have two well known forms [9, 10] and [11]. Quantum error correction conditions [7, 12] play a fundamental role in the quantum error correction theory as a good guidance of devising good quantum error correction codes. Fault-tolerant quantum computations [13, 14, 15, 16, 17] study how to perform reliable quantum computation with the help of quantum error correction codes, and the threshold theorem [18, 19] can be used to evaluate various proposals for fault-tolerant quantum computations and guide physicists to devise reasonable experiments in quantum information and computation.

Quantum computer is a physical system described by quantum mechanics, i.e., its dynamics determined by the Hamiltonian in the Schrödinger equation. The focus of the present paper is to study the Hamiltonian model underlying quantum error correction code. In Kitaev's toric code [14], the Hamiltonian is a linear combination of elements of the stabilizer group, its degenerate ground state represents a quantum error correction code, and there is a gap between its ground state and its first excited state to protect this code from environment noise. Subsequent to the toric code is topological quantum computing [14, 20] in which qubits are anyons (quasiparticles obeying the braid statistics) and quantum gates form unitary braid representations. In condensed matter physics, Abelian or non-Abelian anyons can be created in the fractional quantum Hall effect [21]. Recently, encoding quantum information into a subsystem of a physical system has been found to be a most general method for protecting quantum information from decoherence, and it is summarized in operator quantum error correction [12, 22, 23]. Similar to the Hamiltonian model of the toric code, two examples for how to construct a Hamiltonian model for operator quantum error correction subsystem codes are presented by Bacon [24]. Remarkably, the Hamiltonian model of the toric code [25] on a four dimensional lattice and the Hamiltonian model of the subsystem code [24] on the three-dimensional cubic lattice may be theoretical candidates for self-correcting quantum memory where the robust storage of quantum information is guaranteed by physical properties of the system (i.e., the Hamiltonian and boundary conditions or topology). Besides, Hamiltonian models for self-correcting quantum memory, fault-tolerant quantum computation can be described by a physical system with a time-dependent Hamiltonian [26, 27, 28]. In this paper, however, we will discuss quantum error correction code in a different Hamiltonian formalism from what have been introduced as above. We study quantum states themselves which represent a

quantum error correction code, rewrite them by a unitary transformation on the product basis, and then derive the Hamiltonian to determine this unitary transformation. The stabilizer formalism is not involved in our time-independent Hamiltonian formulation. We will explain our motivation in detail with simple examples in the following sections.

In the literature, there have been many new kinds of quantum error correction approaches, for examples: nonadditive codes [29, 30, 31, 32, 33] beyond the stabilizer formalism, entanglement-assisted quantum error correction [34], topological color codes [35], subsystem codes [36, 37], channel-adapted codes [38, 39, 40], etc. Creating new quantum error correction code is not a primary goal of this paper, whereas revisiting quantum error correction codes from a new point of view is our first point. Quantum error correction code is represented by a unitary basis transformation matrix from the product basis to the entangling basis in terms of quantum error correction code, and this explicitly shows the entanglement nature of quantum error correction codes (i.e., “fight entanglement with entanglement” by Preskill [15, 16]). A possible experiment realization of such a basis transformation matrix can be referred to [41]. We are only concerned about quantum error correction code itself instead of its underlying physical, informational, algebraic, graphical, topological, and geometrical mechanism. On the other hand, we will indeed articulate a new topic called “quantum error correction codes using Greenberger-Horne-Zeilinger (GHZ) states” which include as examples the Shor nine-qubit code [3], amplitude damping channel-adapted codes [38, 39, 40], and nonadditive codes [29, 40]. It is possible to invent a theoretical framework for devising quantum error correction codes using GHZ states based on recognizable properties of GHZ states. GHZ states [42, 43, 44] are usually assumed to be maximally entangled states in various entanglement measure theories [1, 2].

As a remark, our research represents a further development of a recent study presented in [45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56]. We exploit observations and techniques in these articles (especially [52]) to study quantum error correction, which has not been done before to the best of our knowledge. They mainly study interdisciplinary themes arising from quantum information and computation, low dimensional topology [57], the Yang-Baxter equation [58, 59, 60] and (almost-complex) differential geometry. For examples: connections among quantum entanglements, topological entanglements and geometric entanglements as well as *integrable quantum computation* are discussed in [45, 46, 47, 48]; a possible link [52] between quantum error correction and topological quantum computing can be set up via *the Bell matrix*, which forms a unitary braid representation and acts as a unitary basis transformation matrix from the product basis to GHZ states. In the sense of quantum information and computation, these papers articulate *integrable quantum computation* as well as *topological-like (partial topological) quantum computation* [54, 55, 56]: the former is a new approach for quantum computation using *integrable models* [47, 48, 58, 59, 60] constructed via solutions of the quantum Yang-Baxter equation (the Yang-Baxter equation with the spectral parameter); and the latter exploits low dimensional topology [57], unitary braid gates and non-braid gates. In our Hamiltonian formulation, Shor’s code (and its generalization) can be explained as a braiding operator, and the corresponding Hamiltonian

model can be recognized as an integrable model.

We hereby summarize our main result which is new to our knowledge. 1) Quantum error correction code is described in a new Hamiltonian formulation, and the unitary evolution operator at the specific time is a unitary basis transformation from the product basis to the quantum error correction code. 2) In our Hamiltonian formulation, Shor's code and its generalization can be recast as braiding operators. 3) Quantum error correction codes using GHZ states are analyzed as an independent topic for yielding new nonadditive codes or channel-adapted codes. The plan of this paper is organized as follows. In Section 2, with the helpful formalism of GHZ states, Shor's nine-qubit code is described in the Hamiltonian formulation and then explained in the language of the braid group. In Section 3, we discuss quantum error correction codes using GHZ states, study generalized Shor's codes as examples, and make comments on nonadditive codes and channel-adapted codes. Finally, we conclude this paper by presenting interesting problems for further research.

2 Notations, motivations and Shor's nine-qubit code

Notations for most symbols in this paper are introduced, and an overview is made on physical, informational and mathematical properties of GHZ states. Hamiltonian formulations of Shor's nine-qubit code are explored in detail, so that our motivations for the Hamiltonian formulation of quantum error correction codes (without exploiting the stabilizer formalism) can be understood well. Besides, the original motivation takes root in our previous work [52] recasting GHZ states as unitary solutions of the Yang-Baxter equation (or the braid group relation).

2.1 Notations

In quantum information and computation [1, 2], a two dimensional Hilbert space \mathcal{H}_2 over the complex field \mathbb{C} is called a qubit, for example, $\alpha|0\rangle + \beta|1\rangle$, $\alpha, \beta \in \mathbb{C}$ where $|0\rangle$ and $|1\rangle$ form an orthonormal basis of $\mathcal{H}_2 \cong \mathbb{C}^2$ and are usually chosen as eigenvectors of the Pauli operator Z : $Z|0\rangle = |0\rangle$ and $Z|1\rangle = -|1\rangle$. The Pauli matrices have the conventional form,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = ZX = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (1)$$

which denote the bit-flip, phase-flip, and bit-phase flip operation on a qubit.

The symbol \mathbb{I}_2 denotes a 2-dimensional identity operator or $2 \otimes 2$ identity matrix. The notations $A^{\otimes n}$ and $|a\rangle^{\otimes n}$ denote the following n -fold tensor products,

$$A^{\otimes n} = \underbrace{A \otimes \cdots \otimes A}_n, \quad |a\rangle^{\otimes n} = \underbrace{|a\rangle \otimes \cdots \otimes |a\rangle}_n. \quad (2)$$

An n -fold tensor product, in terms of the identity operator \mathbb{I}_2 and Pauli matrices X, Y, Z , has a simpler notation in which the lower index of the Pauli matrix labels its

position in this tensor product, for example, Z_1Z_2 and $X_4X_5X_6X_7X_8X_9$ in a 9-fold tensor product describing the form, respectively,

$$Z_1Z_2 = Z \otimes Z \otimes (\mathbb{1}_2)^{\otimes 7}, \quad X_4X_5X_6X_7X_8X_9 = (\mathbb{1}_2)^{\otimes 3} \otimes X^{\otimes 6}. \quad (3)$$

The symbols M and B are exploited in the entire paper. The M is an anti-Hermitian operator $M^\dagger = -M$ satisfying $M^2 = -Id$ with Id denoting the identity operator, and $B = e^{\frac{\pi}{4}M}$. But M and B will have different presentations depending on where they appear. If M is a k -fold tensor product in terms of Pauli matrices X, Y, Z , then we introduce a notation for an n -fold tensor product M_i by

$$M_i = (\mathbb{1}_2)^{\otimes i-1} \otimes M \otimes (\mathbb{1}_2)^{\otimes n-k-i+1}, \quad 1 \leq i \leq n+1-k \quad (4)$$

where the lower index of M_i can be relabeled for convenience. In [51, 52], the symbol M is called the *almost-complex structure* in the (almost-complex) differential geometry, and it can yield an anti-Hermitian representation of the *extraspecial two-groups*. As $B = e^{\frac{\pi}{4}M}$ is a solution of the Yang–Baxter equation [58, 59] and can generate a unitary braid representation π_n of the Artin braid group \mathcal{B}_n [57], it is called the *Bell matrix*, see [47, 48, 50, 51, 52]. Since unitary braids are used as quantum gates in topological quantum computing [14, 20] and extraspecial two-groups are exploited in quantum error correction [9, 10], the formula $B = e^{\frac{\pi}{4}M}$ suggests there may exist a link between quantum error correction and topological quantum computing, which is the main proposal of our previous paper [52].

2.2 An overview on Greenberger-Horne-Zeilinger states

The GHZ states [42, 43, 44] are a multipartite generalization of the bipartite maximally entangled Bell states, and are defined by various known entanglement measures as maximally entangled. They play crucial roles in both fundamental problems and practical applications of quantum information theory. The Bell theorem [61] for incompatibility between quantum theory and classical deterministic local models is expressed in the form of inequalities (Bell inequalities) among various statistical correlations, whereas the GHZ theorem [42, 43] asserts that the quantum correlations represented by the GHZ states allows us to describe the Bell theorem in terms of equalities and to test the expected incompatibility only by perfect correlations. The GHZ states are the simplest multipartite maximally entanglement sources, and have been widely exploited in the study of quantum information and computation, for examples, multiparty quantum key distributions [62] and quantum teleportation [63]. Besides, GHZ states are often called *cat states* acting as ancillas in fault-tolerant quantum computation [13, 15, 16, 17].

Besides the fundamental importance of GHZ states in quantum physics and information, they have been found to possess (beautiful and deep) topological, algebraic and geometric properties. Aravind [45] observed that there are similarities between the GHZ states and knot configurations [57], by identifying the measurement of a specific state of a particle with cutting the corresponding link component. Furthermore, we describe GHZ states by the Bell matrix $B = e^{\frac{\pi}{4}M}$ which has become a common topic among

quantum error correction, topological quantum computing, the braid group, the Yang–Baxter equation, and (almost-complex) differential geometry [46, 47, 48, 49, 50, 51, 52]. The GHZ states of n -qubit are defined to have the form,

$$\frac{1}{\sqrt{2}}(|s\rangle \pm |\bar{s}\rangle), \quad s = i_1 i_2 \cdots i_n, \quad \bar{s} = \bar{i}_1 \bar{i}_2 \cdots \bar{i}_n \quad (5)$$

where $i_j = 0, 1$, $j = 1, \dots, n$ and $i_j + \bar{i}_j = 1$ with the Abelian addition modulo 2. In [52], the GHZ states of the n -qubit are generated by the exponential function $B = e^{\frac{\pi}{4}M}$ acting on the product basis, for example,

$$\frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}) = B|0\rangle^{\otimes n}, \quad \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} - |1\rangle^{\otimes n}) = B^{-1}|0\rangle^{\otimes n}, \quad (6)$$

where $M = -Y \otimes X^{\otimes n}$ and $B^{-1} = e^{-\frac{\pi}{4}M}$ denotes the inverse of the B matrix. Note that $B = e^{\frac{\pi}{4}M}$ is called *the Bell matrix* only if it is a unitary solution of the Yang–Baxter equation [58, 59] (or forms a unitary braid representation [57]).

2.3 Hamiltonian formulation of Shor’s nine-qubit code

The first quantum error correction code was found by Shor [3], and it describes a logical qubit state in terms of nine-qubit states in the way

$$\begin{aligned} |0\rangle &\rightarrow |0\rangle_L = \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle), \\ |1\rangle &\rightarrow |1\rangle_L = \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \end{aligned} \quad (7)$$

where two GHZ states $\frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle)$ are repeatedly used and hence Shor’s code is also called the repetition code. In the stabilizer formalism [11] for additive quantum error correction codes, the encoded logical qubit $\alpha|0\rangle_L + \beta|1\rangle_L$, $\alpha, \beta \in \mathbb{C}$ is uniquely determined as a common eigenvector with the eigenvalue 1 of the following eight stabilizer operators,

$$Z_1 Z_2, Z_2 Z_3, Z_4 Z_5, Z_5 Z_6, Z_7 Z_8, Z_8 Z_9, X_1 X_2 X_3 X_4 X_5 X_6, X_4 X_5 X_6 X_7 X_8 X_9 \quad (8)$$

which form an Abelian group called the stabilizer group. The logical bit-flip operation \bar{X} and logical phase-flip operation \bar{Z} usually have the form

$$\bar{X} = Z^{\otimes 9}, \quad \bar{Z} = X^{\otimes 9}. \quad (9)$$

In the following, we firstly describe Shor’s nine-qubit code in our Hamiltonian formulation and then present an interpretation in the language of the braid group. Denote the symbol M by $M = -Y \otimes Y \otimes Y$ in this subsection. It is a transition operator between $|000\rangle$ and $|111\rangle$,

$$M|000\rangle = |111\rangle, \quad M|111\rangle = -|000\rangle \quad (10)$$

due to $-Y|0\rangle = |1\rangle$ and $-Y|1\rangle = -|0\rangle$, and it is an anti-Hermitian operator $M^\dagger = -M$ determining $e^{\frac{\pi}{4}M}$ to be a unitary operator,

$$e^{\frac{\pi}{4}M} = \frac{1}{\sqrt{2}}(\mathbb{1}_2^{\otimes 3} + M), \quad M^2 = -\mathbb{1}_2^{\otimes 3}. \quad (11)$$

Furthermore, introduce M_1, M_4, M_7 as

$$M_1 = M \otimes \mathbb{1}_2^{\otimes 6}, \quad M_4 = \mathbb{1}_2^{\otimes 3} \otimes M \otimes \mathbb{1}_2^{\otimes 3}, \quad M_7 = \mathbb{1}_2^{\otimes 6} \otimes M \quad (12)$$

and denote the summation of M_1, M_4, M_7 by $M_t = M_1 + M_4 + M_7$. After some simple algebra, the Shor nine-qubit code has a new compact formulation,

$$|0\rangle_L = e^{\frac{\pi}{4}M_t}|0\rangle^{\otimes 9}, \quad |1\rangle_L = e^{-\frac{\pi}{4}M_t}|0\rangle^{\otimes 9} \quad (13)$$

which leads to the bit-phase flip operation \bar{Y} on the encoded qubit,

$$|0\rangle_L = e^{\frac{\pi}{2}M_t}|1\rangle_L, \quad |1\rangle_L = e^{-\frac{\pi}{2}M_t}|0\rangle_L. \quad (14)$$

Namely, \bar{Y} has the form

$$\bar{Y} = e^{\frac{\pi}{2}M_t} = M_1 M_4 M_7 = -Y^{\otimes 9} \quad (15)$$

where the minus sign in the front of $Y^{\otimes 9}$ partly explains the reason why in the literature the logical bit-flip operator \bar{X} and the logical phase-flip operator \bar{Z} have the formulation (9) because of $\bar{Y} = \bar{Z}\bar{X}$ and $XZ = -ZX$.

Now let us associate the formulation (14) of the Shor nine-qubit code with the Hamiltonian defined by $H = iM_t$. A logical qubit basis, $|0\rangle_L$ and $|1\rangle_L$, is explained as a unitary evolution of the product basis state $|0\rangle^{\otimes 9}$ (or $|1\rangle^{\otimes 9}$), with the unitary evolution operator $U(\theta) = e^{-i\theta H}$ determined by the Hamiltonian H , namely,

$$|0\rangle_L = U\left(\frac{\pi}{4}\right)|0\rangle^{\otimes 9}, \quad |1\rangle_L = U\left(-\frac{\pi}{4}\right)|0\rangle^{\otimes 9} = -U\left(\frac{\pi}{4}\right)|1\rangle^{\otimes 9} \quad (16)$$

which leads that the encoded qubit $\alpha|0\rangle_L + \beta|1\rangle_L$ has a form

$$\alpha|0\rangle_L + \beta|1\rangle_L = U\left(\frac{\pi}{4}\right)(\alpha|0\rangle^{\otimes 9} - \beta|1\rangle^{\otimes 9}), \quad \alpha, \beta \in \mathbb{C}. \quad (17)$$

The Shrödinger equation has the form with the unitary evolutionary solution $U(\theta)$,

$$i\frac{\partial}{\partial\theta}\psi(\theta) = H\psi(\theta), \quad H = -i(Y_1Y_2Y_3 + Y_4Y_5Y_6 + Y_7Y_8Y_9) \quad (18)$$

where the Planck constant $\hbar = 1$, θ is regarded as the time variable and $\psi(\theta)$ represents the wave function in quantum mechanics. The three-body spatially local Hamiltonian H can be simulated on various lattice models, but all of which may be equivalent to a one-dimensional spin chain because only the Y operation is involved in this Hamiltonian. Note that we do not impose any boundary conditions on the Shrödinger equation (18), mainly because they are determined by which type of lattice model to be chosen.

In our Hamiltonian formulation, obviously, Shor's nine-qubit code is not an eigenvector of the Hamiltonian H and is not its ground state. This is essentially different from Hamiltonian models [14][24] respectively for the toric code and subsystem code, which are constructed with the help of the stabilizer formalism. In spite of this fact, we expect our Hamiltonian formulation to be helpful for exploring the topics of how to encode and decode Shor's code, how to perform the fault-tolerant quantum computation with Shor's code, and especially how to construct Hamiltonian model of nonadditive code without using the standard stabilizer formalism. In the next subsection, furthermore, we will recast the Shor nine-qubit code as a braiding operator, and explain the motivation for our Hamiltonian formulation from the different point of view.

2.4 Artin braid group, Yang–Baxter equation and Shor's code

The Artin braid group \mathcal{B}_n [57] has the generators σ_i , $i = 1, \dots, n-1$ satisfying the braid group relations 1) and commutative relations 2):

$$1) \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_i, \quad 2) \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| > 1. \quad (19)$$

In terms of the identity matrix $\mathbb{1}_2$ and a k -fold tensor product M involving Pauli matrices, a unitary braid representation π_n of the Artin braid group \mathcal{B}_n can be constructed in the way,

$$B_i \equiv \pi_n(\sigma_i) = (\mathbb{1}_2)^{\otimes i-1} \otimes e^{\frac{\pi}{4}M} \otimes (\mathbb{1}_2)^{\otimes n-k-i+1}, \quad 1 \leq i \leq n+1-k. \quad (20)$$

where $e^{\frac{\pi}{4}M}$ satisfies the Yang–Baxter equation [58, 59, 52],

$$(G \otimes Id)(Id \otimes G)(G \otimes Id) = (Id \otimes G)(G \otimes Id)(Id \otimes G), \quad (21)$$

where $G = e^{\frac{\pi}{4}M}$ and $Id = \mathbb{1}_2$ and which is a presentation of the braid group relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_i$, see [52] for more details. Besides, the quantum Yang–Baxter equation is defined as the Yang–Baxter equation with the spectral parameter,

$$(G(x) \otimes Id)(Id \otimes G(xy))(G(y) \otimes Id) = (Id \otimes G(y))(G(xy) \otimes Id)(Id \otimes G(x)), \quad (22)$$

where x, y are the spectral parameter and which can be referred to our previous papers [47, 48, 50, 51, 52]. It is well known that in the literature an *integrable model* can be constructed using a solution of the quantum Yang–Baxter equation, see [60].

In Figure 1, the diagram without boxes around crossings is a diagrammatic representation of the Yang–Baxter equation (21) or (22) in which different strands are allowed to represent Hilbert spaces of different dimensions, for example, thin strands acting on \mathcal{H}_2 (i.e., a qubit) and thick strands acting on the tensor product of \mathcal{H}_2 (i.e., at least two qubits). On the other hand, $G = e^{\frac{\pi}{4}M}$ can be explained as a universal quantum gate [64, 46] which transforms a separate state to a maximally entangled state, for example, a GHZ state. The diagram with boxes is hence regarded as a diagrammatical identity between two quantum circuits where universal quantum gates are labeled by boxes with inputs and outputs. In other words, we are presenting two sorts of interpretations on the same diagrammatical object: the one is the Yang–Baxter equation

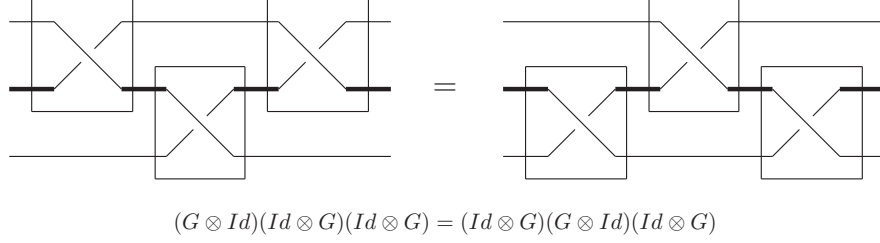


Figure 1: Yang–Baxter equation and quantum circuits

(or the braid group relation) [57, 58, 59, 60] and the other is quantum information and computation [1, 2]. We are trying to build a bridge between quantum information theory and Yang–Baxter equation (or low dimensional topology), which is the underlying motivation of a series of papers [47, 48, 50, 51, 52, 53, 54, 55, 56]. For examples, we have recognized *Werner states* [65] as a sort of rational solutions of the quantum Yang–Baxter equation [53]; in the formulation of GHZ states [51, 52], found mathematical structures including unitary braid representations, extraspecial two-groups and almost-complex structure; described quantum teleportation protocol [63] by a braiding operator [54, 55]; and recast diagrammatical quantum information protocols involving maximally bipartite entangled states as the extended Temperley–Lieb diagrammatical configuration [54, 55, 56]. In the following, we refine the braid group relation from our Hamiltonian formulation of the Shor nine-qubit code.

Choose M to be the form $M = -Y \otimes X \otimes X$ and construct the unitary braid representations B_1, B_4, B_7 by

$$B_1 = e^{\frac{\pi}{4}M} \otimes \mathbb{1}_2^{\otimes 6}, \quad B_4 = \mathbb{1}_2^{\otimes 3} \otimes e^{\frac{\pi}{4}M} \otimes \mathbb{1}_2^{\otimes 3}, \quad B_7 = \mathbb{1}_2^{\otimes 6} \otimes e^{\frac{\pi}{4}M}, \quad (23)$$

and then rewrite Shor’s nine-qubit code into a compact formulation

$$|0\rangle_L = B_1 B_4 B_7 |0\rangle^{\otimes 9}, \quad |1\rangle_L = B_1^{-1} B_4^{-1} B_7^{-1} |0\rangle^{\otimes 9}. \quad (24)$$

The B_1, B_4, B_7 are presentations of the braid group generators, and have a diagrammatic representation, see Figure 2. Every straight strand denotes a two-dimensional Hilbert space \mathcal{H}_2 (i.e., a qubit), and the crossing denotes $e^{\frac{\pi}{4}M}$ and acts on $\mathcal{H}_2^{\otimes 3}$. For convenience, we can assign \mathcal{H}_2 to the first strand of the crossing and $\mathcal{H}_2^{\otimes 2}$ to the second strand. Therefore, we read the Shor nine-qubit code from the braiding configuration as a product of three braids B_1, B_4, B_7 .

On the other hand, in terms of $M_t = M_1 + M_4 + M_7$ where M_1, M_4, M_7 have a form using the convention (3),

$$M_1 = -Y_1 X_2 X_3, \quad M_4 = -Y_4 X_5 X_6, \quad M_7 = -Y_7 X_8 X_9, \quad (25)$$

we obtain the other Hamiltonian formulation of Shor’s code similar to (14) but the logical bit-phase flip operation \overline{Y} given by

$$\overline{Y} = -Y_1 X_2 X_3 Y_4 X_5 X_6 Y_7 X_8 X_9. \quad (26)$$

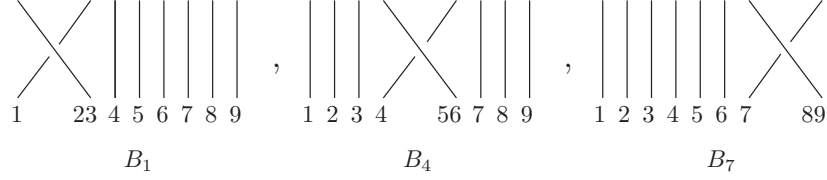


Figure 2: Artin braid group generators for Shor's nine-qubit code

In view of our previous work on *integrable quantum computation* [47, 48, 50, 51, 52], the unitary evolution operator $U(\theta) = e^{-i\theta H}$ with $H = iM_t$ is a solution of the quantum Yang–Baxter equation (22) with the spectral parameter θ , and hence the corresponding Hamiltonian model is an integrable model determined by the solution of the quantum Yang–Baxter equation [58, 59, 60].

As a remark about our unitary braiding description of Shor's nine-qubit code, it provides an interesting example for the observation [52] that there exists a possible link between quantum error correction and topological quantum computing via the unitary braid representation using anti-Hermitian representation of extraspecial two-groups. Besides, it is possible to make connections between our work and a geometric paradigm of Shor's code [66].

3 Quantum error correction codes using GHZ states

Quantum error correction codes using GHZ states are defined as those codes in terms of GHZ states or tensor products of GHZ states, while *generalized Shor's codes using GHZ states* are specified to be *repetition codes* as a straightforward generalization of the Shor nine-qubit code, i.e., only involving GHZ states of the type, $1/\sqrt{2}(|0\rangle^{\otimes n} \pm |1\rangle^{\otimes n})$. They are degenerate quantum error correction codes in which different errors may be corrected by the same correction operation. Besides the Shor nine-qubit code, some nonadditive codes [29, 40] and channel-adapted codes [38, 39, 40] are good examples for quantum error correction codes using GHZ states. The motivation for our articulation of this topic is based on the observation that GHZ states have nontrivial physical, informational, and mathematical properties (see our overview on GHZ states in Subsection 2.2), and these properties are expected to be very helpful in the construction of interesting quantum error correction codes using GHZ states.

In the literature, the triple $[n, k, d]$ of natural numbers represents a quantum error correction code [4]. It is a unitary mapping \mathcal{Q} from $\mathcal{H}_2^{\otimes k}$ to $\mathcal{H}_2^{\otimes n}$ ($k \leq n$), and encodes information of k -qubit quantum states into n -qubit quantum states in order to detect and correct t -qubit errors, where $t = \frac{d-2}{2}$ for d even and $t = \frac{d-1}{2}$ for d odd. Quantum states in this 2^k -dimensional subspace $\mathcal{QH}_2^{\otimes k}$ are called *codewords*. For example, Shor's nine-qubit code is usually denoted by $[9, 1, 3]$ and it can correct any single-qubit errors, while the logical qubit $\alpha|0\rangle_L + \beta|1\rangle_L$, $\alpha, \beta \in \mathbb{C}$ represents a codeword in Shor's code.

In this section, for simplicity, we denote quantum error correction codes $[n, k, d]$ by

$[n, k]$, and focus on whether or not they are able to correct arbitrary single-qubit errors, i.e., $t = 1$. For example, the Shor code is denoted by $[9, 1]$. Note that we emphasize our perspectives of constructing *quantum error correction codes using GHZ states* and describe them in our Hamiltonian formulation, but leave the systematic construction of procedures for encoding, error syndrome measurement, error correction and decoding of these codes elsewhere.

3.1 Generalized Shor's codes using GHZ states

First of all, we introduce symbols and names necessary for constructing generalized Shor's codes using GHZ states. The *type* of a GHZ state $\frac{1}{\sqrt{2}}(|s\rangle \pm |\bar{s}\rangle)$ (5) is specified by the number of its qubits and the sign \pm . Generalized Shor's codes exploit different types of GHZ states to encode one qubit into codewords of N -qubit. The same GHZ states form a *block* in the codeword, and different blocks correspond to orthogonal subspaces of $\mathcal{H}_2^{\otimes N}$. The symbol I is the number counting types of GHZ states (or the number of blocks) in the codeword; i denotes the i -th type of GHZ state (or i -th block) in the codeword, $i = 1, \dots, I$; q_i counts the number of qubits in the i -th type of GHZ state; n_i is the repetition number of the i -th type of GHZ state in the i -th block of the codeword; $B = \sum_{i=1}^I n_i$ counts the number of GHZ states in a codeword; $N = \sum_{i=1}^I n_i q_i$ is the total number of qubits in the codeword.

For example, the Shor nine-qubit code exploits GHZ states of three-qubit and repeatedly use them three times in a codeword: $I = 1$, $q_1 = 3$, $n_1 = 3$, $B = 3$ and $N = 9$. On the other hand, $B = 3$ is the number counting all phase-flip single-qubit errors (i.e., Z -errors), and $2N = 18$ counts all bit-flip and phase-flip errors (i.e., X -errors and Y -errors). Hence Shor's code satisfies the quantum Hamming bound condition,

$$2^t(1 + B + 2N) \leq 2^N \quad (27)$$

where $t = 1$, and which suggests the Shor code as a degenerate code and able to correct some two-qubit errors. Now let us discuss an application of the above bound inequality at $t = 1$. Obviously, the lowest bound for N has to be $N \geq 5$.

At $N = 5$, we can either construct a logical qubit using GHZ states of five-qubit,

$$|0\rangle_L = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes 5} + |1\rangle^{\otimes 5}), \quad |1\rangle_L = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes 5} - |1\rangle^{\otimes 5}) \quad (28)$$

where $B = 1$, $n_1 = 1$ and $q_1 = 5$, or have a logical qubit in terms of Bell states (i.e., GHZ states of two-qubit) and GHZ states of three-qubit,

$$|0\rangle_L = \frac{1}{2}(|0\rangle^{\otimes 2} + |1\rangle^{\otimes 2})(|0\rangle^{\otimes 3} + |1\rangle^{\otimes 3}), \quad |1\rangle_L = \frac{1}{2}(|0\rangle^{\otimes 2} - |1\rangle^{\otimes 2})(|0\rangle^{\otimes 3} - |1\rangle^{\otimes 3}) \quad (29)$$

where $B = 2$, $n_1 = n_2 = 1$, $q_1 = 1$ and $q_2 = 1$.

At $N = 6$, we can construct a logical qubit in terms of GHZ states of six-qubit (or four-qubit or three-qubit) or Bell states, for examples,

$$|0\rangle_L = \frac{1}{2\sqrt{2}}(|0\rangle^{\otimes 2} + |1\rangle^{\otimes 2})(|0\rangle^{\otimes 2} + |1\rangle^{\otimes 2})(|0\rangle^{\otimes 2} + |1\rangle^{\otimes 2}),$$

$$|1\rangle_L = \frac{1}{2\sqrt{2}}(|0\rangle^{\otimes 2} - |1\rangle^{\otimes 2})(|0\rangle^{\otimes 2} - |1\rangle^{\otimes 2})(|0\rangle^{\otimes 2} - |1\rangle^{\otimes 2}). \quad (30)$$

where $B = 3$, $n_1 = 3$ and $q_1 = 3$.

In view of our Hamiltonian formulation of the Shor nine-qubit code, generalized Shor's codes have the following Hamiltonian formulation,

$$|0\rangle_L = e^{\frac{\pi}{4} \sum_{i=1}^I M_t^{(i)}} |0\rangle^{\otimes N}, \quad |1\rangle_L = e^{-\frac{\pi}{4} \sum_{i=1}^I M_t^{(i)}} |0\rangle^{\otimes N} \quad (31)$$

where the Hamiltonian $H^{(i)} = \sqrt{-1} M_t^{(i)}$ determines how to yield the i -th type of GHZ state in the i -th block of the codeword and has the form

$$H^{(i)} = \sqrt{-1} \sum_{j=1}^{n_i} M_j^{(i)}. \quad (32)$$

Different $M_j^{(i)}$ are commutative with each other because the corresponding subspace of each type of GHZ state in the codeword is orthogonal with those for other types. Note that the lower indices of $M_j^{(i)}$ are different from that exploited in the Hamiltonian formulations of the Shor nine-qubit code in Subsections 2.3 and 2.4.

Note on a link between the cat code $[n, 1]$ and quantum error correction codes using GHZ states. The cat code $[n, 1]$ has the form $|0\rangle_L = |0\rangle^{\otimes n}$ and $|1\rangle_L = |1\rangle^{\otimes n}$ leading to

$$\frac{1}{\sqrt{2}}(|0\rangle_L \pm |1\rangle_L) = e^{\pm \frac{\pi}{4} M} |0\rangle_L, \quad |0\rangle_L + |1\rangle_L = M(|0\rangle_L - |1\rangle_L) \quad (33)$$

in which M can take a form $M = -Y \otimes X^{\otimes n}$ or other possible forms. Besides, it is interesting to compare *generalized Shor's codes using GHZ states* with *generalized Shor's subsystem codes* [36], because they are constructed under completely different motivations and terminologies.

3.2 Comments on channel-adapted codes and nonadditive codes

As has been shown in Subsection 2.2, GHZ states are maximally entangled multipartite quantum states and possess (simple and deep) algebraic, topological and geometric properties. These properties have been used to explore connections among quantum information and computation, the Yang-Baxter equation, low dimensional topology and differential geometry, see [52]. On the other hand, GHZ states have been exploited to yield quantum error correction codes [29, 38, 39, 40]. Some of them [38, 39] can be described in the stabilizer formalism, i.e., additive codes, but many of them [29, 40] are nonadditive codes which do not have classical analogues and do not satisfy the quantum Hamming bound (27). Note that nonadditive codes using GHZ states are called *self-complementary codes* in [29, 40]. Some codes using GHZ states (either additive or nonadditive) are channel-adapted quantum error correction for the amplitude damping channel [38, 39, 40]. However, it remains an open problem how to create nonadditive codes using GHZ states in a unified theoretical framework, though interesting examples

have been found. We expect that algebraic (or topological or geometrical) properties of GHZ states [52] are helpful for solving this problem. This is the motivation for our articulating quantum error correction codes using GHZ states.

As examples, we analyze two codes [38, 39] devised for the amplitude damping channel in our Hamiltonian formalism. They may be helpful for seeking a theoretical framework underlying nonadditive codes using GHZ states. The first one [38] is the code $[4, 1]$ having a logical qubit spanned by

$$|0\rangle_L = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle), \quad |1\rangle_L = \frac{1}{\sqrt{2}}(|0011\rangle + |1100\rangle). \quad (34)$$

This code can be rewritten as the unitary basis transformation $e^{-i\frac{\pi}{4}H}$,

$$(|0\rangle_L, |1\rangle_L) = e^{-i\frac{\pi}{4}H}(|0000\rangle, |0011\rangle) \quad (35)$$

where $iH = Y \otimes X^{\otimes 3}$ is called the almost-complex structure and the Bell matrix $B = e^{-i\frac{\pi}{4}H}$ forms a unitary braid representation, see [52]. Note that the Bell matrix B can generate all the GHZ states of four-qubit from the product basis, i.e., it has all the information of GHZ states of four-qubit. In this sense, the Hamiltonian formulation of the code $[4, 1]$ suggests: it appears to work only with two GHZ states of four-qubit, but in fact it involves all other GHZ states of four-qubit through the Bell matrix B . This may be one of the reasons why it violates the quantum Hamming bound (27) but works well for the amplitude-damping channel. The second example is the code $[6, 2]$ which has the form in the Hamiltonian formulation,

$$(|00\rangle_L, |01\rangle_L, |10\rangle_L, |11\rangle_L) = e^{-i\frac{\pi}{4}H}(|000000\rangle, |001001\rangle, |000110\rangle, |110000\rangle), \quad (36)$$

where $H = -iY \otimes X^{\otimes 5}$. A similar analysis can be made: the unitary evolution operator $e^{-i\frac{\pi}{4}H}$ has the information of all GHZ states of six-qubit, and hence the code $[6, 2]$ actually encodes two-qubit information into the entire Hilbert space spanned by $2^6 = 64$ GHZ basis states instead of its four-dimensional subspace. In this sense, we may understand why a 12-dimensional subspace can be encoded into the GHZ states of 8-qubit and protected by a nonadditive code [40].

Furthermore, we discuss how to protect a one-dimensional space by quantum error correction codes using GHZ states. The quantum Hamming bound condition (27) at $t = 0$ has a perfect solution in terms of $B = 1$, $N = 3$ saturating the bound, i.e., $(1 + 1 + 2 \times 3) \leq 2^3$. Let us encode the one-dimensional space $|\psi\rangle$ using a GHZ state of three-qubit,

$$|\psi\rangle_L = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) = e^{-i\frac{\pi}{4}H}|0\rangle^{\otimes 3}, \quad H = -iY \otimes Y \otimes Y. \quad (37)$$

There are three bit-flip errors, one phase-flip error and three bit-phase flip errors giving rise to the other seven GHZ states of three-qubit,

$$X_1|\psi\rangle_L, X_2|\psi\rangle_L, X_3|\psi\rangle_L, Z_1|\psi\rangle_L, Z_1X_1|\psi\rangle_L, Z_2X_2|\psi\rangle_L, Z_3X_3|\psi\rangle_L, \quad (38)$$

which form an orthonormal basis of $\mathcal{H}_2^{\otimes 3}$ together with $|\psi\rangle_L$. They can be recast in our Hamiltonian formulation

$$\begin{aligned} & (Id, X_1, X_2, X_3, Z_1, Z_1X_1, Z_2X_2, Z_3X_3)|\psi\rangle_L \\ &= e^{-i\frac{\pi}{4}H}(|000\rangle, |011\rangle, |110\rangle, -|111\rangle, -|100\rangle, -|010\rangle, -|001\rangle) \end{aligned} \quad (39)$$

where four minus signs can be absorbed by rescaling the phase-flip Z -error. Besides, we can exploit the standard procedures of encoding, error syndrome measurement, correcting and decoding devised for the stabilizer codes [11].

4 Concluding remarks and outlook

Motivated by the problem how to construct a Hamiltonian model for nonadditive quantum error correction code, we suggest a new Hamiltonian formulation of quantum error correction code without appealing to its stabilizer formalism (if it exists). As a remark, in Kitaev's fault tolerant schemes [14], thermal effect may not be easily surmountable² which can destroy the expected fault-tolerance, see [67], and hence seeking for new Hamiltonian models of quantum error correction codes remains a fundamental problem in the current research of quantum information and computation.

In our Hamiltonian formulation, the unitary evolution operator at a specific time is a unitary basis transformation matrix from the product basis to the quantum error correction code, and we explain this by studying examples including Shor's nine-qubit code and its generalization. Remarkably, as this basis transformation matrix is a solution of the Yang-Baxter equation, the quantum error correction code can be explained as a braiding operator, and the Hamiltonian model is an integrable model determined by the solution of the quantum Yang-Baxter equation. On the other hand, we articulate the topic called *quantum error correction codes using GHZ states*, in which new nonadditive codes and channel-adapted codes may be constructed with the help of beautiful properties of GHZ states.

There still remain many interesting problems about our work in this paper. 1) Devise a theoretical framework for describing quantum error correction codes in the Hamiltonian formulation without involving the stabilizer formalism. Our typical examples are mainly based on algebraic properties of GHZ states. 2) Study physical properties of our Hamiltonian models and explore their applications to quantum error correction. As a comparison, quantum error correction code in the Hamiltonian model based on the stabilizer formalism is its degenerate ground state, and there is a gap between this ground state and its first excited state. 3) Study experimental realizations of these Hamiltonian models on optical lattices or atomic spin lattices [68, 69]. The unitary basis transformation from the product basis to GHZ states can be experimentally performed in view of the work [41]. 4) Study applications of these Hamiltonian models to develop a theoretical framework for *integrable quantum computation* or *topological-like quantum computation*. Note that the unitary evolution operator in the Hamiltonian

²The author thanks Zohar Nussinov for the email correspondence on this point.

model for a quantum error correction code using GHZ states can be chosen as a solution of the quantum Yang–Baxter equation. 5) About *quantum error correction codes using GHZ states*, there exist at least three types of questions to be answered: devise a general framework for yielding nonadditive codes; study quantum circuits for encoding, error correction and decoding; explore fault-tolerant quantum computation with the help of properties of GHZ states, for example, channel-adapted fault-tolerant quantum computation for the amplitude damping channel.

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